nucleon has a *total angular momentum quantum number* $j = l \pm \frac{1}{2}$ except in case $l = 0$ when j has only one value $= l + \frac{1}{2} = \frac{1}{2}$. Thus each energy level splits up into two sub-levels with $j \uparrow\uparrow = l + \frac{1}{2}$ ($\vec{s}$ and $\vec{l}$ parallel) and $j \uparrow\downarrow = l - \frac{1}{2}$ ($\vec{s}$ and $\vec{l}$ anti-parallel). The sub-level $j + \frac{1}{2}$ has a lower energy than the sub-level $j - \frac{1}{2}$. Thus, the $j + \frac{1}{2}$ state represents a more lightly bound nucleon. The separation between two sub-levels increases rapidly with l and for $l \geq 4$, the two sub-levels lie in different shells. Each sub-level can have a maximum of $(2j + 1)$ nucleons of the same kind *i.e.*, it can accommodate $(2j + 1)$ protons and $(2j + 1)$ neutrons.

We have already introduced a quantum number similar to the principal quantum number for the electronic orbit which characterises the radial part of the nuclear wave function and is denoted by n. The value of $n = 1, 2, 3, \ldots$etc. Thus, a nuclear quantum state is represented by three quantum numbers n, l and j. In accordance with Pauli's exclusion principle no two identical nucleons can be in the same quantum state *i.e.*, have the same value of all the three quantum numbers. Pauli's principle is thus, applied to the protons and neutrons separately and two protons or two neutrons with spins $+\frac{1}{2}$ and $-\frac{1}{2}$ are allowed in each level. The neutron levels are equally spaced whereas the spacing between the proton levels increases gradually as we move from the ground level to the higher levels. This is due to the Coulomb force of repulsion between the protons.

For example, for $l = 0$, $j = l + \frac{1}{2} = \frac{1}{2}$ and the number of nucleons in the level $= (2j + 1) = 2$.

For $l = 1, j = 3/2$ and $\frac{1}{2}$ and the number of nucleons in the two sub-levels are $\left(2 \times \frac{3}{2} + 1\right) = 4$ and $\left(2 \times \frac{1}{2} + 1\right) = 2$ repectively. The total number of nucleons in this level $= 4 + 2 = 6$ giving the progressive total of the number of nucleons in the nucleus $= 8$. In this way, the completed energy levels or shells and the corresponding total number of nucleons can be predicted.

As already stated the energy levels corresponding to each value of l for the nucleon are represented by a spectroscopic notation similar to be electrons as shown,

$l = 0, 1, 2, 3, 4, 5, \ldots$

Spectroscopic notation $= s, p, d, f, g, h, \ldots$

The nucleons are designated by writing their n values followed by the spectroscopic notation giving their l values with their j values written as subscript and the number of nucleons required to complete a sub-level written as superscript.

Under this scheme $1s$ state has $l = 0$. Taking $s = \frac{1}{2}$; $j = l + \frac{1}{2} = \frac{1}{2}$ ($j = l - \frac{1}{2}$ not allowed).

This state is now represented as $(1s_{1/2})^2$ as the number of nucleons in this state $= (2j + 1) = 2$. $1p$ state has $l = 1$ and therefore $j = l + \frac{1}{2} = \frac{3}{2}$ and $j = l - \frac{1}{2} = \frac{1}{2}$. This state is therefore split into two states denoted as $(1p_{3/2})^4$ [as it has $\left(2 \times \frac{3}{2} + 1\right) = 4$ nucleons] and $(1p_{1/2})^2$ [as it has $\left(2 \times \frac{1}{2} + 1\right) = 2$ nucleons].

The quantum numbers n, l and j, and designation of energy levels according to above notation in order of the energy of the levels upto magic number 82 is shown in **Table I.**

It should be noted from the table that level $(2s_{1/2})^2$ lies between $(1d_{3/2})^4$ and $(1d_{5/2})^6$. Similarly level $(2p_{1/2})^4$ lies between $(1f_{5/2})^6$ and $(1f_{7/2})^8$ wheres level $(2p_{1/2})^2$ lies between $(1g_{9/2})^{10}$ and $(1f_{5/2})^6$ and so on.

Achievements (Predictions). 1. Magic numbers. The shell model explains the existence of *magic numbers*. By making a refrence to the above table and by adding the nucleons from below the shell closure occurs at neutron or proton number 2, 8, 20, 50, 82 and 126 which correspond to magic numbers.

n	l	j	Designation	Level	$2j+1$	Total no. of protons or neutrons in the closed shell	Progressive Total
1	5	$\frac{11}{2}$	$(1h_{11/2})^{12}$ ...	...	12		
3	0	$\frac{1}{2}$	$(3s_{1/2})^2$ ...	...	2		
2	2	$\frac{3}{2}$	$(2d_{3/2})^4$ ...	...	4	32	
2	2	$\frac{5}{2}$	$(2d_{5/2})^6$ ...	...	6		
1	4	$\frac{7}{2}$	$(1g_{7/2})^8$ ...	...	8		82
1	4	$\frac{9}{2}$	$(1g_{9/2})^{10}$ ...	...	10		
2	1	$\frac{1}{2}$	$(2p_{1/2})^2$ ...	...	2		
1	3	$\frac{5}{2}$	$(1f_{5/2})^6$ ...	...	6	30	
2	1	$\frac{3}{2}$	$(2p_{3/2})^4$ ...	...	4		50
1	3	$\frac{7}{2}$	$(1f_{7/2})^8$ ...	...	8		
1	2	$\frac{3}{2}$	$(1d_{3/2})^4$ ...	...	4		
2	0	$\frac{1}{2}$	$(2s_{1/2})^2$ ...	...	2	12	
1	2	$\frac{5}{2}$	$(1d_{5/2})^6$ ...	...	6		20
1	1	$\frac{1}{2}$	$(1p_{1/2})^2$ ...	...	2		
1	1	$\frac{3}{2}$	$(1p_{3/2})^4$ ...	...	4	6	8
1	0	$\frac{1}{2}$	$(1s_{1/2})^2$ ...	...	2	2	

Table I

2. Spin. The shell model successfully explains the *ground state spins* and *magnetic moments* of the nuclei. The neutrons and protons with opposing spin *pair off* so that their mechanical and magnetic moments cancel. It is, therefore, the odd or left proton or neutron which could not be paired that contributes to the spin and the magnetic moment of the nucleus as a whole.

The total angular momentum quantum number I, generally referred to as the spin quantum number, is obtained by adding the j-values of individual nucleons in it, subject to the usual quantum conditions.

For example, ${}_1H^2$ has $Z = 1$ *i.e.*, one proton and $N = A - Z = 1$ neutron. Each nucleus will, therefore, be in the ground state $I = 0$ and will have a j-value $\frac{1}{2}$ for either. The j's of the proton and the neutron are oriented parallel to each other so that $I = 1$ which agrees with the experimentally observed value of spin. For ${}_1H^3$ and ${}_2He^3$ nuclei we have two nucleons of the same kind which are paired off and one nucleon of the other kind with j value $\frac{1}{2}$. Thus the I value of each of these two

nuclei is $\frac{1}{2}$. Similarly for ${}_6C^{13}$, the nuclear configuration is $(1s_{1/2})^{2+2}\ (1p_{3/2})^{4+4}\ (1p_{1/2})^{1}$ and hence $I = \frac{1}{2}$ the j-value for the last unpaired neutron. For ${}_8O^{17}$ we have $(1s_{1/2})^{2+2}\ (1p_{3/2})^{4+4}\ (1p_{1/2})^{2+2}\ (1d_{5/2})^{1}$ and $I = 5/2$ the j-value for the last unpaired neutron.

3. Magnetic moment. The observed nuclear magnetic moments are also explained in terms of j-values of the odd nucleon in the nucleus.

The magnetic moment of a nucleus depends upon the configuration whether it is an *even-even* nucleus, *even-odd* or *odd-even* nucleus. For an *even-even* nucleus the nuclear magnetic moment is zero. For *odd-even* (or *even-odd*) nucleus, the nuclear magnetic moment depends upon whether the last unpaired nucleon is a proton or a neutron.

The magnetic moment associated with orbital angular momentum vector $\vec{l}$ is given by

$$\mu_l = g_l \vec{l} = g_l \sqrt{l(l+1)}$$

Where g_l is the g-factor for orbital motion and l is the orbital quantum number.

The magnetic moment associated with spin angular momentum vector $\vec{s}$ is given by

$$\mu_s = g_s \vec{s} = g_s \sqrt{s(s+1)}$$

Where g_s is the g-factor for spin motion and s is the spin quantum number.

Therefore, total magnetic moment μ = sum of the components of vectors $g_l \vec{l}$ and $g_s \vec{s}$ along $\vec{j}$ -vector. Hence in units of $\hbar$.

$$\mu = g_l \sqrt{l(l+1)}\cos(\vec{l}\,.\,\vec{j}) + g_s \sqrt{s(s+1)}\cos(\vec{s}\,.\,\vec{j})$$

Now $$\cos(\vec{l}\,.\,\vec{j}) = \frac{l^2 + j^2 - s^2}{2\,\vec{l}\,.\,\vec{j}} = \frac{\vec{l}\,.\,\vec{l} + \vec{j}\,.\,\vec{j} - \vec{s}\,.\,\vec{s}}{2\,\vec{l}\,.\,\vec{j}}$$

$$= \frac{l(l+1) + j(j+1) - s(s+1)}{2\sqrt{l(l+1)}\sqrt{j(j+1)}}$$

Similarly, $$\cos(\vec{s}\,.\,\vec{j}) = \frac{s^2 + j^2 - l^2}{2\,\vec{s}\,.\,\vec{j}} = \frac{s(s+1) + j(j+1) - l(l+1)}{2\sqrt{s(s+1)}\sqrt{j(j+1)}}$$

$\therefore$ $$\mu = g_l \sqrt{l(l+1)}\,\frac{l(l+1) + j(j+1) - s(s+1)}{2\sqrt{l(l+1)}\sqrt{j(j+1)}} + g_s \sqrt{s(s+1)}\,\frac{s(s+1) + j(j+1) - l(l+1)}{2\sqrt{s(s+1)}\sqrt{j(j+1)}}$$

or $$\mu = g_l\,\frac{l(l+1) + j(j+1) - s(s+1)}{2\sqrt{j(j+1)}} + g_s\,\frac{s(s+1) + j(j+1) - l(l+1)}{2\sqrt{j(j+1)}}$$

Magnetic moment of a single nucleon. For a single particle, spin $s = \frac{1}{2}$ in units of $\hbar$ and there are two possible cases

(*i*) $\vec{l}$ is parallel to $\vec{s}$. This is known as *stretch case.*

For such a case $$j = l + s = l + \tfrac{1}{2}$$

$$\therefore \quad \mu = g_l\left(j - \tfrac{1}{2}\right) + \tfrac{1}{2}g_s$$

(*ii*) $\vec{l}$ is anti-parallel to $\vec{s}$. This is known as *Jack-knife case.*

For such a case $j = l - s = l - \frac{1}{2}$

$$\therefore \quad \mu = g_l\left(\frac{j}{j+1}\right)\left[\left(j + \tfrac{3}{2}\right) - \tfrac{1}{2}g_s\right]$$

For single nucleons

$g_l = 1$ and $g_s = 5.58$ for proton.

$g_l = 0$ and $g_s = -3.82$ for neutron.

Thus we find that the odd proton has a positive magnetic moment due to spin whereas the odd neutron has a negative magnetic moment. Thus, the spins and magnetic moments of isotopic nuclei with odd A differing in their A-values by 2 are the *same* since the two added neutrons are paired off and do not have any effect on the single odd nucleon which contributes to the spin and magnetic moment of the nucleus. For example, ${}_{47}Ag^{107, 109}$ and ${}_{55}Cs^{133, 135, 137}$ isotopes have the same value of nuclear spin and magnetic moment.

4. Stability. The very great stability and high binding energy is explained on the basis of *closed shells.* For example, for ${}_2He^4$ the first shell is completely filled up by protons as well as neutrons (2 each) so that $I = 0$. Further the closure of the shell gives it a very high binding energy of 28.3 MeV. Thus the α-particle ${}_2He^4$ nucleus is a very stable configuration. The case of ${}_8O^{16}$ is also similar with first two shells for neutrons and protons completely filled up with 2 and 6 nucleons *i.e.* a total of 8 nucleons respectively.

In adding nucleons to the nucleus the proton and the neutron levels should be equally filled up for maximum stability. Hence for small values of mass number A, the number of neutrons is very nearly equal to the number of protons. As the number of nucleons becomes large, the most stable nuclides will again be formed when the lowest energy levels available to protons and neutrons are filled up. As the spacing between proton levels goes on incresing, the number of neutrons will be greater than the number of protons.

5. Nuclear isomerism. The phenomenon of *nuclear isomerism i.e.*, existense of isobaric, isotopic nuclei in different energy states observed in nuclei with odd A values lying between 39 to 49, 69 to 81, 111 to 125 has been explained on the shell model due to the large difference in the nuclear spins of the isomeric states of these nuclei because they have their A values close to the magic numbers required to complete the various nuclear shells.

Failures and limitations. 1. The shell model does not predict correct values of I in case of certain nuclei. For example, it predicts $I = 5/2$ for ${}_{11}Na^{23}$ whereas the correct value is $\frac{3}{2}$.

2. The four stable nuclei ${}_1H^2$, ${}_3Li^6$, ${}_5B^{10}$, ${}_7N^{14}$ do not fit in the scheme of this model.

3. This model cannot explain the observed first excited states in even-even nuclei at energies much lower than those expected from single particle excitation.

4. This model could not explain the observed large 'quadruple moment' of odd A nuclei, particularly those which have their nucleon number far away from the magic numbers. The shell model assumes the nucleus to be spherical and a new model known as *collective model* has been developed which assumes that the last nucleon of an odd-A nucleus interacts with the remaining even number of nucleons, deviating it from spherical symmetry. Thus a nucleus with all filled shells like ${}_8O^{16}$ has a spherical shape but a nucleus with odd A like ${}_8O^{17}$ acquires a non-spherical deformity and hence has a *quadrupole moment.*

The quadrupole moment for a given I is given by

$$Q = -\frac{2I-1}{2I+2}(r_{r.m.s.})^2$$

where $I = l \pm s$ and $(r_{r.m.s.})^2$ is the mean square radius of the orbit. For uniform charge distribution $(r_{r.m.s.})^2 = \frac{3}{5}R_c^2$ where R_c is the Coulomb radius = $1.2\,A^{1/3}$ Fermi.

Q. 23.10. Why are even-even nuclei most stable ? (*G.N.D.U.* 2002)

OR

Why even-even nuclei are the most common whereas odd-odd nuclei are least common ? (*P.U.* 1999)

Ans. Even-even and odd-odd nuclei. Even-even nuclei *i.e.*, nuclei having an even number of protons and even number of neutrons are more stable than odd-odd nuclei *i.e.*, nuclei having an odd number of protons and odd number of neutrons.

According to liquid drop model, nucleons have a tendency to exist in pairs. The pairing energy is given by $E_p = (\pm, 0)a_p A^{-3/4}$ where a_p is a constant having a value – 34 MeV for even Z, even A nuclei thereby adding to the stability of such nuclei.

This is also clear from the shell model of the nucleus. According to Pauli's exclusion principle a single energy sub-level can have a maximum of two nucleons, one with spin up (↑) and the other with spin down (↓). For such nuclei excited states can be produced only by breaking up a pair of nucleons which requires a very high 'pairing energy'. Thus in even-even nuclei only completed sub-shells are present which results in greater stability. On the other hand, an odd-odd nucleus contains incomplete subshells for both kind of nucleons which give rise to lesser stability As a result even-even nuclei are most common and there are about 164 such stable nuclei out of which six even-even nuclei ${}_8O^{16}$, ${}_{12}Mg^{24}$, ${}_{14}Si^{28}$, ${}_{20}Ca^{40}$, ${}_{22}Ti^{48}$, and ${}_{26}Fe^{56}$ form about 80% of earth's crust. For the same reason odd-odd nuclei are least common, there being only six such stable nuclei. Except for ${}_7N^{14}$ all others are very rare.

Q. 23.11. Solve Schrodinger's wave equation for the ground state of deutron. How is the quadrupole moment of the deutron explained ? (*Luck.U.* 2001)

Ans. Schrondinger's wave equation for the ground state of deutron. Schrondinger's wave equation is given by

$$\nabla^2\psi + \frac{2m}{\hbar^2}[E - V]\psi = 0 \quad \ldots (i)$$

where E = Total energy of the system

V = Potential energy of the system due to forces acting between the two nucleons

m = reduced mass

In terms of spherical polar co-ordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Substituting the value of ∇^2 in Eq. (*i*), we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$+ \frac{2m}{\hbar^2}[E - V(r,\theta,\phi)]\psi = 0 \quad \ldots (i)$$